

Analytic Solutions for Viscous Plumes and Thermals

John Lister & Robert Whittaker
ITG, DAMTP, University of Cambridge

Introduction

Plume 'heads' and 'tails' are an important ingredient of mantle convection, giving rise to flood-basalt provinces and hot-spot volcanism.

We consider Boussinesq convection in Stokes flow

$$\nu \nabla^2 \mathbf{u} = \nabla p - g\alpha(T - T_0)\mathbf{e}_z \quad \nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T$$

with constant ν, κ, g, α . The buoyancy $b = g\alpha(T - T_0)$.

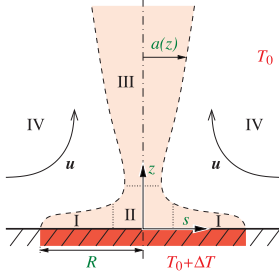
New asymptotic solutions have been obtained for

- (1) a steady plume rising from a constant-temperature heat source
- (2) unsteady rise of a hot pulse, or 'thermal', with constant total buoyancy.

(1) A STEADY PLUME

Q: What is the plume and boundary-layer structure above a hot source of radius R ?

Q: How does the Nusselt number depend on Rayleigh number $Ra = g\alpha\Delta TR^3/\nu\kappa$ for $Ra \gg 1$?



- Solution involves matching: (I) an inflowing boundary layer above the source; (II) a short turn-round region; (III) a slender rising plume; (IV) the induced external flow.

- Boundary conditions:

$$\text{At } \infty: \quad b \rightarrow 0, \quad \mathbf{u} \rightarrow 0$$

$$\text{On } z = 0: \quad b = g\alpha\Delta T \quad (s < R), \quad \frac{\partial b}{\partial z} = 0 \quad (s > R),$$

either $\mathbf{u} = 0$ (rigid) or $u_z = 0, \frac{\partial u_s}{\partial z} = 0$ (free-slip)

Region I: Boundary layer above source

- The plume induces an inward flow (Region IV) over the source
- The boundary condition (rigid/free-slip) has a big influence on the strength and importance of this flow near the boundary:

For a free-slip b.c. the plume-induced flow in region IV dominates the locally driven flow

For a rigid b.c. the flow driven by the local buoyancy dominates the plume-induced flow

- Solution gives the total buoyancy and mass fluxes in the thermal boundary layer.

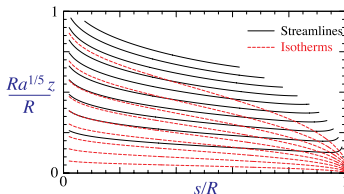


Figure: Numerical solution for the thermal boundary layer above a rigid boundary. This matches to the turn-round region as $s \rightarrow 0$ and to the external flow as $Ra^{1/5}z \rightarrow \infty$

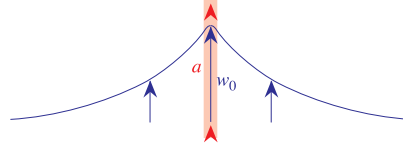
Region II: Turn-round

- The fluid spends too little time in this region for diffusion to act significantly. The heat and mass fluxes are transmitted unaltered from the boundary layer to the base of the plume.

Region IV: External flow

- The external flow sees a line force $F(z) = \int b dA$ spread over a thickness $a(z) \ll z$. From slender-body theory the velocity within the plume is nearly uniform and given by

$$w \sim w_0(z) = \frac{F}{2\pi\nu} \ln \frac{z}{a} \quad \text{for } s \leq a$$



Region III: Slender plume

- The plume is narrow with typical radius $a(z) \ll z$. Vertical diffusion is negligible since $\partial b/\partial z \ll \partial b/\partial s$
- Using a stream function ψ , one can show that the vertical buoyancy flux

$$B \equiv \int w b dA = \int b d\psi \text{ is constant}$$

and the buoyancy-weighted vertical mass flux

$$Q(z) \equiv 2 \int_0^\infty b \psi d\psi \bigg/ \int_0^\infty b d\psi = Q(0) + 4\kappa z$$

where $B(0)$ and $Q(0)$ are found from regions I and II.

The buoyancy force and plume thickness are then given by

$$F(z) \equiv \int b dA \sim \frac{B}{w_0} \quad \text{and} \quad a^2(z) = \frac{Q(z)}{w_0(z)}$$

Final scalings

- Putting the calculations all together (for $Ra \gg 1$)

$$Nu \equiv \frac{B}{g\alpha\Delta T \kappa R} \sim 2.90 Ra^{1/5} \quad (\text{rigid})$$

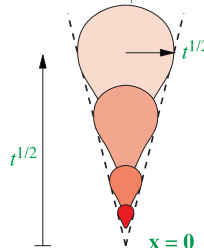
$$Nu \sim \left(\frac{64\pi Ra}{3 \ln Ra} \right)^{1/3} \left(1 - \frac{2 \ln \ln Ra}{3 \ln Ra} \right) \quad (\text{free-slip})$$

- In the plume $w_0(z) \propto (Ra Nu \ln \frac{z}{a})^{1/2}$ varies little with z
- $a(z)$ is approximately constant for $z/R \ll Nu$ and $\propto z^{1/2}$ for $z/R \gg Nu$

(2) AN UNSTEADY THERMAL

Q: How does a finite heat pulse with total buoyancy force $\mathcal{F} = g\alpha \int (T - T_0) dV$ rise and diffuse in Stokes flow?

Q: How does the solution structure depend on the Rayleigh number $Ra = \mathcal{F}/\nu\kappa$ for $Ra \gg 1$?



- The equations support a similarity solution with lengths $\sim t^{1/2}$: by diffusion the size of the pulse scales

like $(\kappa t)^{1/2}$ and the buoyancy scales like $(\kappa t)^{-3/2}$; thus the Stokes rise velocity is proportional to $t^{-1/2}$ and, remarkably, the rise distance is thus also like $t^{1/2}$.

- Define similarity variables

$$\xi = \frac{x}{(4\kappa t)^{1/2}} \quad \mathbf{u}(\mathbf{x}, t) = \left(\frac{\kappa}{4t} \right)^{1/2} \mathbf{U}(\xi)$$

$$g\alpha[T(\mathbf{x}, t) - T_0] = \frac{\mathcal{F}}{(4\kappa t)^{3/2}} \Theta(\xi) \quad p(\mathbf{x}, t) = \frac{\nu}{4t} P(\xi)$$

to obtain the similarity equations

$$\nabla^2 \mathbf{U} = \nabla P - Ra \Theta \mathbf{e}_\zeta, \quad \nabla \cdot \mathbf{U} = 0$$

$$-6\Theta + (\mathbf{U} - 2\xi) \cdot \nabla \Theta = \nabla^2 \Theta, \quad \int \Theta dV = 1$$

which are solved numerically.

Numerical solutions

For $Ra = O(1)$ diffusion is dominant, the thermal has close to a Gaussian distribution, and rises slightly from the origin.

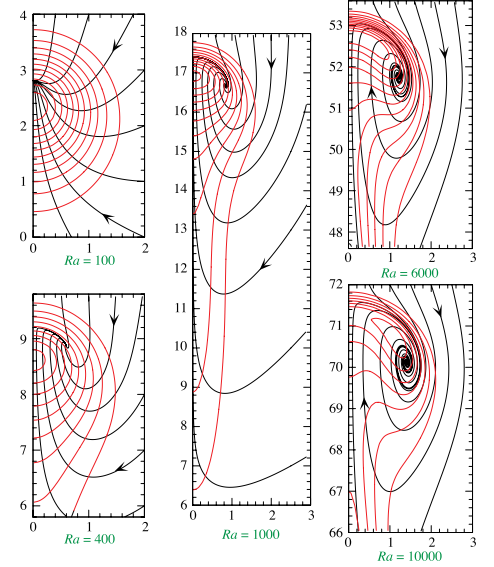


Figure: Numerical solution for the self-similar rise of an axisymmetric thermal pulse. Temperature contours are in red. The material pathlines (in black) are given by $\mathbf{U} - 2\xi$ in the similarity variables.

The solutions for $Ra \gg 1$ (above) show:

- A recirculating head and a long tail
- But most of heat is in the tail not the head
- The rate of rise $\sim (Ra \ln Ra)^{1/2}$ is driven by the tail and is not $\sim Ra^{3/4}$ as proposed by Griffiths (1986)

Slender-body approximation

An asymptotic solution based on slender-body theory for the tail agrees with the numerical scalings with Ra and with the form of \mathbf{U} and Θ .

$$\mathbf{U} \sim (-\rho, 2\zeta), \quad \Theta \sim \frac{12\zeta e^{-3\rho^2/2}}{Ra \ln Ra}$$

for $0 < \zeta < (Ra \ln Ra / 3\pi)^{1/2}$

References

- Griffiths 1986 Thermals in extremely viscous fluids, including the effects of temperature dependent viscosity. *J. Fluid Mech.* **166**, 139–159.
- Whittaker & Lister 2006a,b Steady axisymmetric creeping plumes above a planar boundary. Part I: A point source & Part II: A distributed source. *J. Fluid Mech.* (in press).
- Whittaker & Lister 2006c The self-similar rise of a buoyant thermal in Stokes flow. *J. Fluid Mech.* (in preparation).