High-frequency self-excited oscillations in collapsible tube flows

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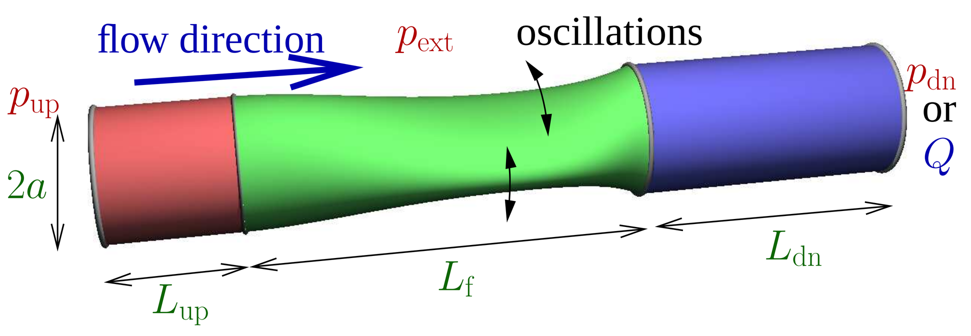
1 Introduction and Overview

Aims, Motivation and Methodology

- Examine an instability of flow through a tube with an elastic wall.
- Motivated by observed self-excited oscillations in Starling resistor experiments (see below), and flow in larger blood vessels.
- A recent review of work on collapsible tube and channel flows is provided by Heil & Jensen (2003)
- Jensen & Heil (2003) studied an instability in 2D channel flow in a high-tension rapid-oscillation regime. We show the same instability mechanism (outlined below) is present in 3D tubes.
- We use a combination of numerical and asymptotic methods to examine the stability criteria and gain insight into the underlying physical mechanisms. We provide theoretical predictions for the stability boundary, growth rates and mode shapes.
- We consider separately the fluid flow (prescribed wall oscillations) and the mechanics of the wall (prescribed transmural pressures) before coupling the two to address the full problem.

Starling Resistor Setup

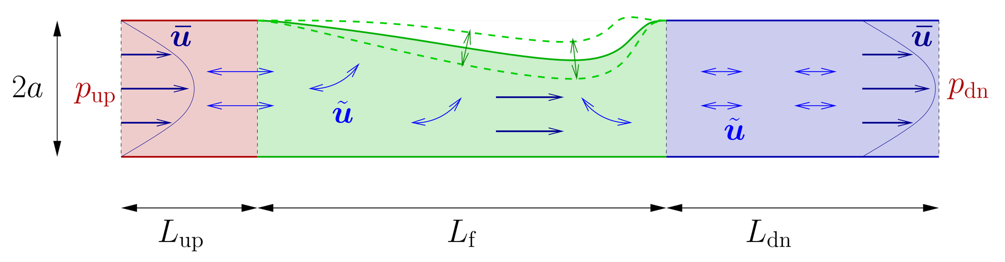
A flexible tube section of length L_f is pinned at both ends to rigid sections of lengths L_{up} and L_{dn}. The total tube length is $L = L_{up} + L_f + L_{dn}$, typical width $2a$, and wall thickness s. The tube wall has bending stiffness K and is stretched axially by a force F. There is an external pressure P_{ext}.



The tube is filled with a fluid of density ρ and dynamic viscosity μ. A steady axial flow of typical scale $\mathscr{U}$ is induced by applying either a steady pressure difference $p_{up} - p_{dn}$ between the ends, or a steady flux Q at the downstream end. The elastic section of wall undergoes oscillations with amplitude d, and frequency $\omega = 2\pi/T$.

2 Instability Mechanism

Mechanism found by Jensen & Heil (2003) for 2D channel flow. We have shown it to operate in 3D tubes too.



Consider energy budget with pressure boundary conditions:

Energy to feed instability = KE Inflow − KE Outflow − Dissipation − Work done at tube ends

- Wall oscillations lead to changes in cross-sectional area.
- This drives an oscillatory axial 'sloshing' flow. Inertial impedance in the rigid sections is proportional to the length, so a greater amplitude occurs in the shorter section.
- The time-averaged KE flux at the tube ends is dominated by the background mean flow, which cancels between the two ends. Then there is a contribution at each end proportional to the square of the amplitude of the sloshing flow there.
- Hence a shorter upstream section ($L_{up} < L_{dn}$) results in a net input of kinetic energy to the fluid inside the system
- Energy can be lost through dissipation and by work done against the pressure at the tube ends.
- Find that work done is $1/3$ of additional KE input, so $2/3$ of this energy input is left over.
- If the dissipation is low enough and there is a large difference in the lengths of the rigid sections, then there will be energy available to feed the instability.

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WHITTAKER, R. J. *et al.* 2009*c* A rational derivation of a tube law from shell theory. In Preparation.

3 Asymptotic Regime

- **Womersley number**:
$$\alpha^2 = \frac{\rho a^2}{\mu T} \gg 1$$
Oscillatory flow has an inviscid core with viscous Stokes layers of thickness $O(\alpha^{-1})$ near the walls.
- **Strouhal number**:
$$St = \frac{a}{\mathscr{U} T} \gg 1$$
Nonlinear inertia absent at leading order.
- **Oscillation amplitude**:
$$\Delta = \frac{d}{a} \ll \alpha^{-1} \ll 1$$
Allows linearisation of boundary conditions.
- **Tube aspect ratio**:
$$\ell = \frac{L}{a} \gg 1$$
Allows long-wavelength approximations to be used.
- **Tube wall thickness**:
$$\theta = \frac{s}{a} \ll 1$$
Allows wall mechanics to be modelled using shell theory.
- **Dimensionless axial tension**:
$$\mathscr{F} = \frac{aF}{2\pi K \ell^2} = O(1)$$
Axial curvature effects comparable with azimuthal bending effects.

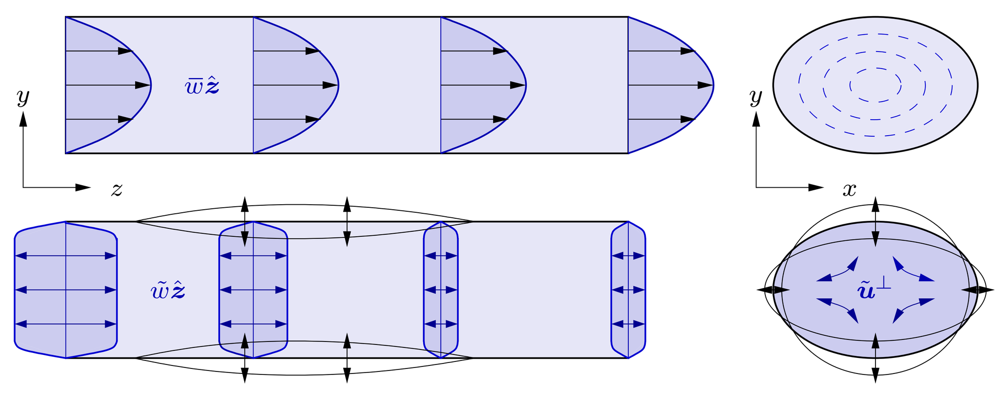
4 Fluid Mechanics (Prescribed Oscillations)

We derive an asymptotic solution to the Navier–Stokes equations for flow through a flexible tube, subject to prescribed high-frequency long-wavelength small-amplitude oscillations of the tube wall.

- Assume a Newtonian fluid, and use the dimensionless Navier–Stokes equations, with velocity $\boldsymbol{u} = \boldsymbol{u}^\perp + \ell w \hat{\boldsymbol{z}}$:

$$\nabla_\perp \cdot \boldsymbol{u}^\perp + \frac{\partial w}{\partial z} = 0\,,$$
$$\frac{\partial \boldsymbol{u}^\perp}{\partial t} + \frac{1}{\ell St}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}^\perp = -\ell^2 \nabla_\perp p + \frac{1}{\alpha^2}\left(\frac{\partial^2 \boldsymbol{u}}{\partial z^2} + \frac{1}{\ell^2}\nabla_\perp^2 \boldsymbol{u}\right),$$
$$\frac{\partial w}{\partial t} + \frac{1}{\ell St}(\boldsymbol{u}\cdot\nabla) w = -\frac{\partial p}{\partial z} + \frac{1}{\alpha^2}\left(\frac{\partial^2 w}{\partial z^2} + \frac{1}{\ell^2}\nabla_\perp^2 w\right).$$

- Assume flow remains laminar, and boundary layers remain attached. Steady component of flow is Poiseuille flow. Oscillatory component is plug flow with boundary-layer structure near wall.



- For asymptotic analysis, decompose variables in multiples of the fundamental oscillation frequency ω, and expand as power series in $\epsilon = \max\{\alpha^{-1}, (\ell St)^{-1}, \ell^{-2}\}$ and $\delta = \Delta/\epsilon$:

$$\boldsymbol{u} = \bar{\boldsymbol{u}}_0 + \delta^2 \bar{\boldsymbol{u}}_2 + \ldots + \lambda\delta\left(\tilde{\boldsymbol{u}}_{00} + \epsilon\tilde{\boldsymbol{u}}_{01} + \epsilon^2\tilde{\boldsymbol{u}}_{02} + \ldots\right)e^{i\omega t} + \lambda\epsilon\delta^2\left(\hat{\boldsymbol{u}}_{00} + \ldots\right)e^{2i\omega t} + \ldots$$

- At leading order, oscillatory pressure and axial velocity are uniform in each cross-section. Continuity and axial momentum:

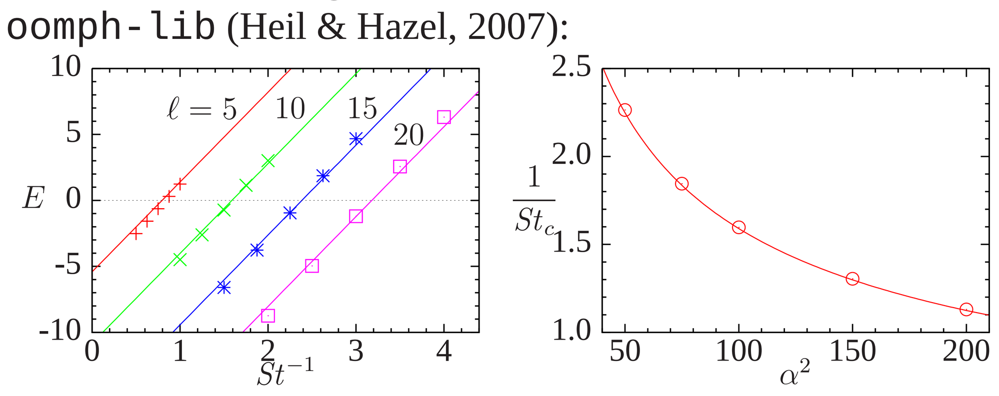
$$\frac{d}{dz}\left(A_0\,\tilde{w}_{00}\right) + i\omega\tilde{A} = 0\,, \qquad i\omega\,\tilde{w}_{00} = -\frac{d\tilde{p}_{00}}{dz}\,. \tag{1}$$

where the cross-sectional area is $A(z,t) = A_0(z) + \tilde{A}(z)e^{i\omega t}$. Have Poisson problem for flow and pressure in each cross-section.

- Solve for leading order flow and first order corrections. Compute energy budget, and find the mean (dimensionless) rate of working E by the fluid on the tube wall.
- Associate $E > 0$ with instability, since more energy is being extracted from the mean flow than is dissipated by the oscillations. Critical inverse Strouhal number St_c^{-1} when $E = 0$.
- Expression for energy in an axially uniform tube with a prescribed flux at the downstream end:

$$E = 2\pi\,|\mathbb{V}(0)|^2\left(\frac{1}{St} - \frac{1}{St_c}\right), \tag{2}$$
$$\frac{1}{St_c} = \frac{\ell}{\alpha}\frac{\pi^{3/2}C_0}{A_0^2\,|\mathbb{V}(0)|^2}\int_0^1 |\mathbb{V}(z)|^2\,dz\,.$$

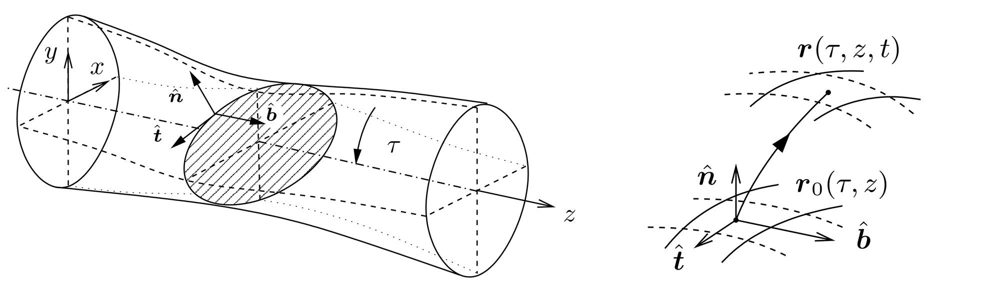
where $\mathbb{V}(z)e^{i\omega t}$ is the volume change between z and the downstream end, and C_0 is the circumference in the undeformed state.
We find excellent agreement with numerical simulations using `oomph-lib` (Heil & Hazel, 2007):



- Also have results for axially non-uniform tubes and the case where the axial flow is driven by a pressure drop. Full details can be found in Whittaker *et al.* (2009*a*,*b*).

5 Solid Mechanics (Tube Law)

We derive an approximate relationship between the transmural pressure $P_{tm}(z)$ and changes in cross-sectional area $A(z)$ for small-amplitude long-wavelength deformations of an initially elliptical elastic-walled tube.



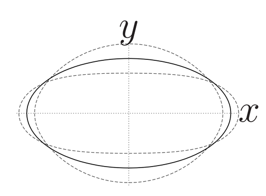
- Parameterise wall with Lagrangian coordinates (τ, z), and triad $(\hat{\boldsymbol{n}}, \hat{\boldsymbol{t}}, \hat{\boldsymbol{b}})$ aligned with surface. Describe deformation by (ξ, η, ζ) such that material initially at dimensionless position $\boldsymbol{r}_0$ moves to

$$\boldsymbol{r} = \boldsymbol{r}_0(\tau,z) + \frac{\Delta}{h}\Big(\xi(\tau,z)\hat{\boldsymbol{n}} + \eta(\tau,z)\hat{\boldsymbol{t}}\Big) + \frac{\Delta}{\ell}\zeta(\tau,z)\,\hat{\boldsymbol{b}}\,,$$

where $h(\tau)$ is the scale factor for the elliptical coordinates.
- We use shell theory to model wall mechanics (Flügge, 1972).
- Long-wavelength ($\ell \gg 1$) and scaled tension ($\mathscr{F} = O(1)$) regime means that the transmural pressure P_{tm} is balanced at leading order by a combination of azimuthal bending and axial curvature forces. Scale of deformations given by $\Delta \sim a^3 P_{tm}/K$. Wall effectively inextensible in azimuthal direction.
- Eliminate ξ and ζ to obtain linear system for $\eta(\tau, z)$:

$$\mathscr{L}_\tau(\eta) - \frac{\partial^2}{\partial z^2}\mathscr{J}_\tau(\eta) = P_{tm}$$

- Deformations of the tube cross-section are dominated by a single mode shape:

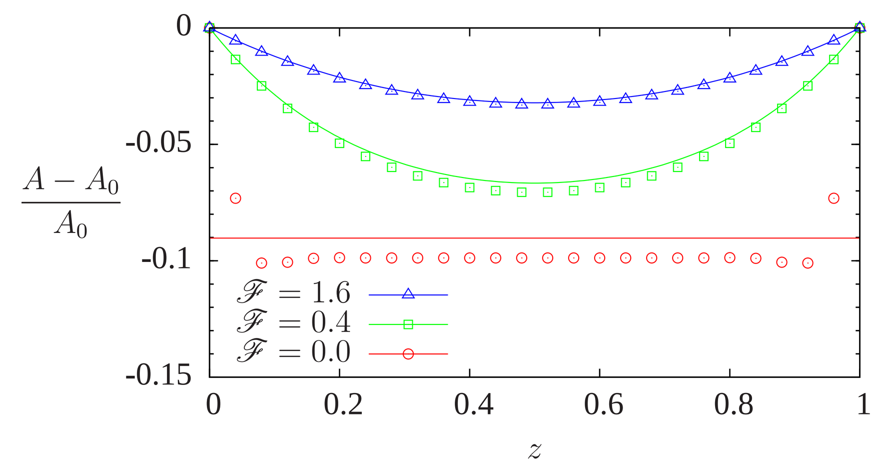


$$\xi(\tau,z) = b_1(z)\,(3 - 4\cosh 2\sigma_0 \cos 2\tau + \cos 4\tau)\,,$$
$$\eta(\tau,z) = 2b_1(z)\,\sinh 2\sigma_0 \sin 2\tau\,.$$

- Truncate to this single mode, and note $A - A_0 \propto b_1(z)$ to obtain:

$$P_{tm} = k_0(A - A_0) - k_2\mathscr{F}\frac{d^2}{dz^2}(A - A_0) \tag{3}$$

in dimensionless variables, where k_0 and k_2 are numerically computed constants related to the geometry of the undeformed tube.
- Good agreement with numerical computations (`oomph-lib`; Heil & Hazel, 2007) for $\ell = 100$, $\theta = 0.05$, and a uniform transmural pressure:



- Further details in Whittaker *et al.* (2009*c*).

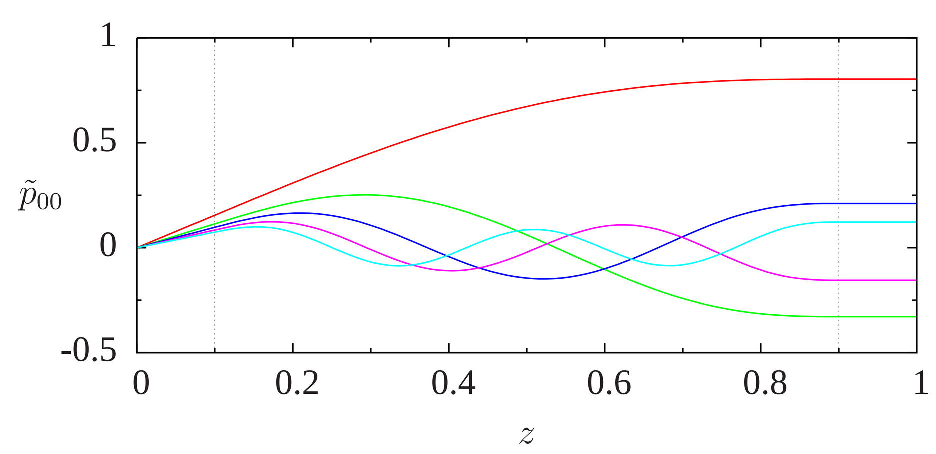
6 Fluid–Structure Interaction (Normal Modes)

Combine fluid and solid mechanics work to consider full FSI problem. Oscillatory normal modes are computed, together with their frequencies and growth rates.

- Combine (1) and (3), and assume oscillatory pressure $\tilde{p}_{00}$ dominates transmural pressure P_{tm}. Eliminate $\tilde{w}_{00}$ and $\tilde{A}$ to form single equation for dimensionless $\tilde{p}_{00}(z)$ in the flexible section:

$$k_2\mathscr{F}\frac{d^4\tilde{p}_{00}}{dz^4} - k_0\frac{d^2\tilde{p}_{00}}{dz^2} - \frac{\omega^2}{A_0}\tilde{p}_{00} = 0\,.$$

- In rigid sections $\tilde{A} = 0 \Rightarrow \tilde{p}_{00}''(z) = 0$.
Match by imposing $[\tilde{p}_{00}] = [\tilde{p}_{00}'] = 0$ at joins.
- Boundary conditions at tube ends:
$\tilde{p}_{00} = 0$ for pressure condition, $\tilde{p}_{00}' = 0$ for flux condition.
- Solve to obtain natural frequencies ω and normal mode shapes:



- Using the energy result (2), growth rate given by

$$\Omega = \frac{\pi}{2A_0}\left(\frac{|\tilde{p}_{00}'(0)|^2}{\ell St\int_0^1 |\tilde{p}_{00}'(z)|^2\,dz} - \frac{(2\omega)^{1/2}}{\alpha}\right).$$

- Fundamental mode has lowest frequency and highest growth-rate.
- Comparison with numerical simulations ongoing.