Fluid–Structure Interaction: Shell Theory and 3D Flow Instability in Collapsible Tubes

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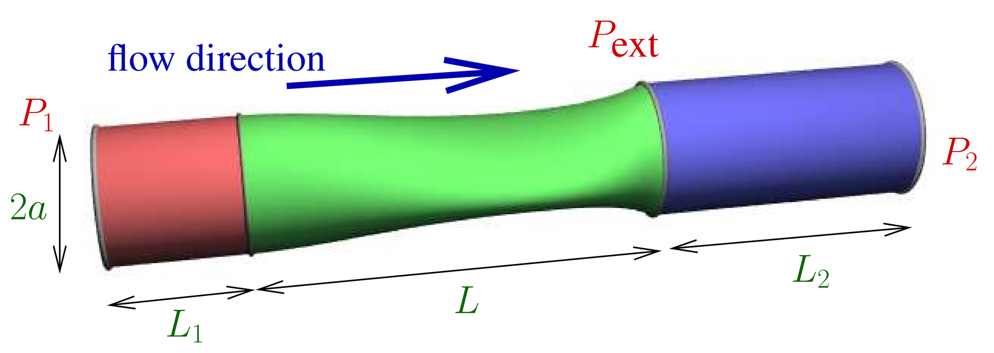
1 Introduction and Overview

Applications and Motivation

- Aim to examine the instabilities of axial flow through an elastic-walled tube. Experiments in such systems develop spontaneous oscillations of the elastic membrane in various regimes.
- Jensen & Heil (2003) studied the instabilities in a 2D channel in a high-tension / fast oscillation regime. In particular, we wish to see if the same instability mechanism is present in 3D tubes.
- Motivated by observed oscillations in Starling resistor experiments (and partly, in turn, by blood flow in larger arteries and veins).
- Use a combination of numerical and asymptotic methods to examine the stability criteria and gain insight into the underlying physical mechanisms.
- A recent review of work on collapsible tube and channel flows is provided by Heil & Jenson (2003)

Model Setup and Boundary Conditions

A simple setup is used, motivated by a Starling resistor: A flexible tube section of length L, pinned at both ends to two rigid sections of lengths L_1 and L_2, and subject to an external pressure P_{ext}. The tube is filled with a fluid of density ρ and dynamic viscosity μ. A length a characterises the cross-sectional width, and the flexible wall has thickness h.



A steady axial flow $\bar{\boldsymbol{u}}$ of typical scale U_0 is induced either by an applied pressure gradient along the tube or by a fixed flux condition at one end. The elastic section of wall is supposed to undergo oscillations with amplitude $d = \epsilon a$, and frequency $\omega = 2\pi/T$.

Tube Collapse Under Steady Flow

Increasing the external pressure can cause the elastic section to partially collapse, reducing the cross-sectional area. Axial tension will reduce this collapse, and the viscous pressure drop along the section (due to the mean flow) means that the collapse will be more pronounced in the down-stream half.

These effects lead to interesting coupled fluid–structure problems: computing the shape of the collapsed tube, and also the resulting pressure–flux relationships for steady flow. For example: Luo & Pedley (1995); Hazel & Heil (2003); Marzo *et al.* (2005)

Forced vs. Free Oscillations

- Primarily interested in the instability, together with the conditions for it to exist, and an understanding of the mechanisms.
- Consider first periodic oscillations of a flexible wall, and calculate the induced fluid flow.
- Examine the net energy transfer to (work done on) the wall. If positive then that particular mode is unstable, and hence we should expect a similar unstable mode to be able to grow spontaneously in an unforced situation.
- As well as these implications, such results also provide further direct validation for numerics, without having to specify any particular wall elasticity model.

2 Parameters and Scaling

Key Dimensionless Groups

- Oscillation amplitude: $\epsilon = \dfrac{d}{a} \ll 1$ (small displacements)
- Womersley number: $\alpha^2 = \dfrac{\rho a^2}{\mu T} \gg 1$ (unsteady Reynolds number)
- Strouhal number: $St = \dfrac{a}{U_0 T} \gg 1$ (time scale ratio)
- Tube aspect ratio: $\ell = \dfrac{L}{a} \gg 1$ (long tube)
- Tube wall thickness: $\delta = \dfrac{h}{a} \ll 1$ (thin walls)

Scaling Regime

Work with ϵ as primary small parameter. Adopt the following scalings for the other dimensionless groups:

- $R^2 = \epsilon^2 \alpha^2 = O(1)$ — viscous boundary layers have thickness $\alpha^{-1} = O(\epsilon)$, same order as oscillation amplitude.
- $\lambda = \epsilon \, \ell \, St = O(1)$ — axial oscillatory velocity is comparable with the axial mean flow velocity.
- $\beta^{-1} = \epsilon \, \ell^2 = O(1)$ — the tube is slender enough to take advantage of a long wavelength expansion.

3 Fluid Mechanics

- Assume a Newtonian fluid, and use the Navier–Stokes equations:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}\right) = -\boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

- Assume flow remains laminar. Then expect an inviscid core with viscous Stokes layers near the walls. Assume these boundary layers remain attached (valid at least for small amplitudes).
- For asymptotic analysis, decompose variables in multiples of the fundamental oscillation frequency ω, and expand in powers of the dimensionless amplitude ϵ:

$$\boldsymbol{u} = \boldsymbol{u}_{00} + \boldsymbol{u}_{01}\,e^{i\omega t} + \epsilon\left(\boldsymbol{u}_{10} + \boldsymbol{u}_{11}\,e^{i\omega t} + \boldsymbol{u}_{12}\,e^{2i\omega t}\right) + \epsilon^2\left(\boldsymbol{u}_{20} + \boldsymbol{u}_{21}\,e^{i\omega t} + \dots\right) + O(\epsilon^3)$$

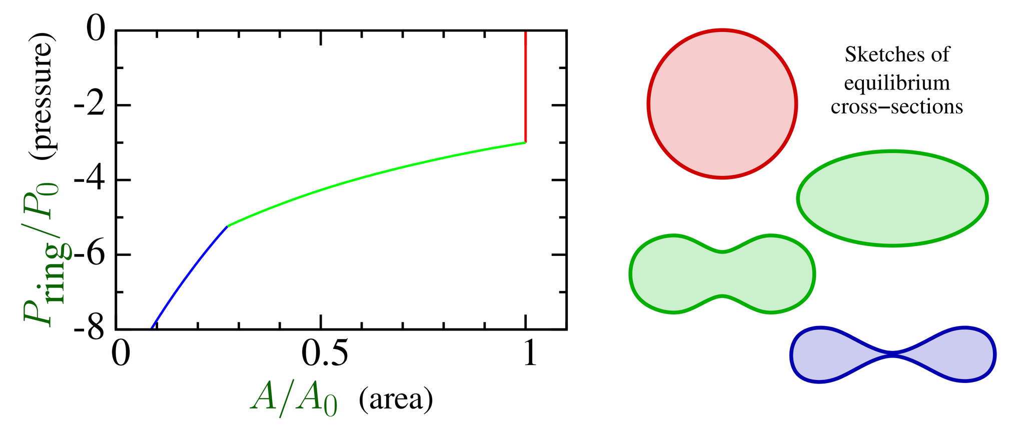
4 Solid Mechanics (Shell Theory)

- Model tube wall as a thin shell using shell theory (see Wempner & Talaslidis, 2003) and assuming linear elastic behaviour. Can derive equations for force balance on the shell mid-plane:

$$\nabla_\mu\left[\gamma^{\mu\nu}\boldsymbol{A}_\nu + \nabla_\nu\left(\tau^{\mu\nu}\hat{\boldsymbol{N}}\right)\right] + \boldsymbol{F} + \nabla_\mu\left[G^\mu \hat{\boldsymbol{N}}\right] = \boldsymbol{0}\,.$$

Notation: γ stress, τ torsion, $\boldsymbol{A}_\mu$ in-plane basis vectors, $\hat{\boldsymbol{N}}$ surface normal, $\boldsymbol{F}$ external force per unit area, G^μ external moment per unit area.

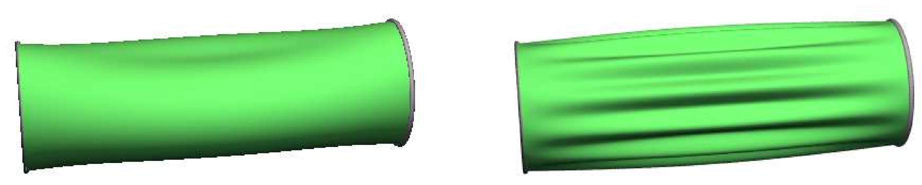
- For thin shells, the bending stiffness is much less than the extensional stiffness, which leads to interesting stability problems.
- Behaviour of an inextensible ring under uniform transmural pressure: buckles in $n = 2$ azimuthal mode (Flaherty *et al.*, 1972).



- Behaviour often extrapolated in an *ad hoc* fashion to form a 'tube law', a pressure–area relationship, with additional dependence on axial variation. Simple example:

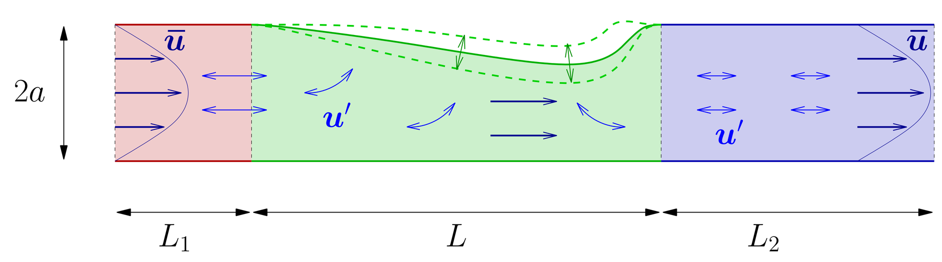
$$P = P_{\text{ring}}(A) - k\frac{\partial^2 A}{\partial z^2}$$

- However, such simple tube laws don't take account of the shape of the partially collapsed tube, and this is vital for accurate representations and modelling.
- Buckling of long tube — axial tension can effect which azimuthal wavenumber n is the most unstable. Higher tension and/or a shorter tube leads to larger n.



5 2D Model and Mechanism

Asymptotic analysis and numerical computations by Jensen & Heil (2003). Apply pressure boundary conditions at two ends, and use rigid sections of different lengths.



Examine instability by considering energy budget:

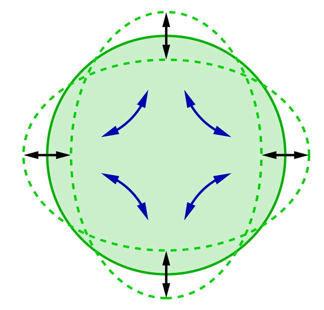
Energy to Wall = KE Inflow − KE Outflow − Dissipation

A net transfer of energy to the wall allows the oscillation amplitude to grow.

- Oscillation amplitude ϵ leads to $O(\epsilon)$ cross-sectional area change.
- This drives an oscillatory axial 'sloshing' flow. Inertial impedance in the rigid sections is proportional to the length, so a greater amplitude occurs in the shorter section.
- The time-averaged KE flux at the tube ends is dominated by the background mean flow, which cancels between the two ends. Then there is a contribution at each end proportional to the square of the amplitude of the sloshing flow there.
- Hence a shorter upstream section results in a net input of kinetic energy to the fluid inside the system
- Energy can only be lost through dissipation in the Stokes layers and/or transfer by net work on the wall.
- Keep dissipation low enough and have a large enough difference in the rigid section lengths, and energy is transfered to the wall.

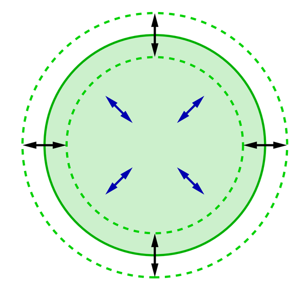
6 Extensions to 3D

Circular Tube with $n = 2$ Oscillations



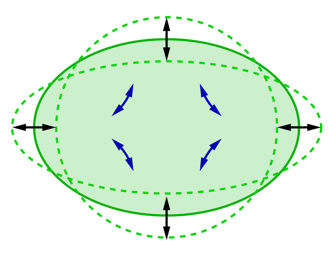
$A = A_0 + O(\epsilon^2)$ so only a weak axial sloshing flow is driven. See Heil & Waters (2006) for cross-sectional flow. Axial flow weaker than in the 2D case, and so any possible energy transfer to the wall only comes in at a higher order in ϵ.

Circular Tube with Axisymmetric Oscillations



$A = A_0 + O(\epsilon)$, but unphysical and little difference from 2D case. Calculations for forced oscillations indicate that that the 2D mechanism still works, and energy can be transfered to the wall.

Elliptical Tube with $n = 2$ Oscillations

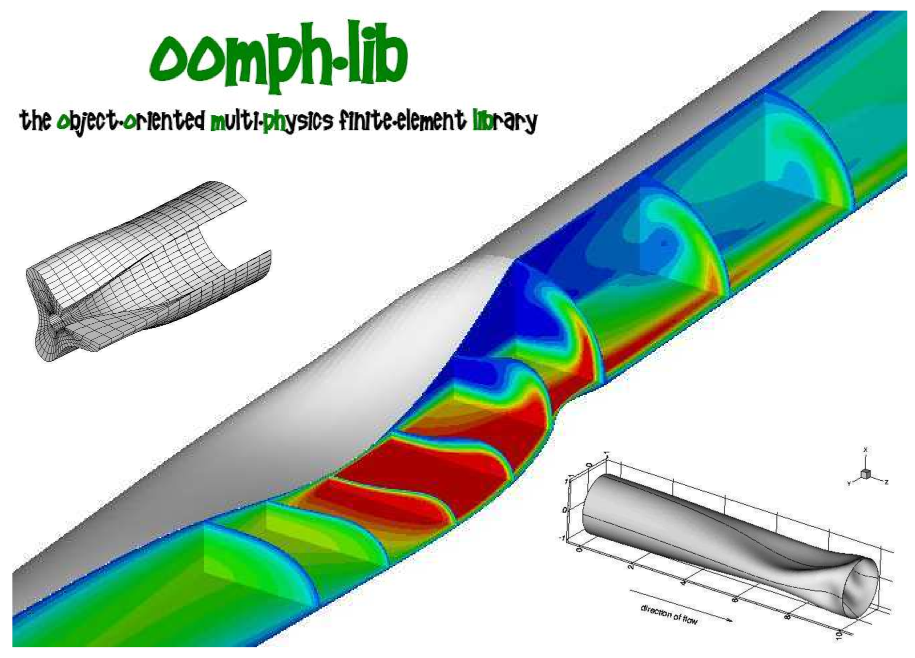


$A = A_0 + O(\epsilon)$ but we have the additional effect of stagnation point flow in cross-sections. Currently working through the details to see if the cross-sectional flow affects the mechanism. Analysis is complicated by the need to use elliptical coordinates.

- Decompose wall deformations into a double Fourier series in the azimuthal and axial variables.
- Assumed slenderness of the tube means cross-sections decouple from one another to some extent.
- Solve Poisson problem for the inviscid cross-sectional flow in the core, forced by the normal wall velocity.
- Mass continuity determines cross-sectionally uniform axial sloshing flow and required axial pressure gradients.
- Solve a moving boundary-layer problem to match tangential velocities between the core and walls.
- Repeat at next order, and look at the energy budget.

7 Numerics: Oomph-lib

An open-source object-oriented multi-physics finite-element library, initially developed by Matthias Heil and Andrew Hazel. Further details and download at: `http://www.oomph-lib.org/`.



In collaboration with Matthias Heil and Jonathan Boyle at Manchester, we aim to use numerical computations to further examine the stability of tube flows, with the asymptotic results providing mutual checks on accuracy.

Acknowledgements

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